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ON THIN FILM DELAMINATION GROWTH IN A CONTRACTING CYLINDER†

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Abstract—The problem of a thin layer delaminating from a contracting cylindrical substrate is considered. An analytical solution is obtained along with associated conditions for growth of the delamination, and specific numerical results corresponding to layers at both the interior and exterior surfaces of the substrate are presented.

INTRODUCTION

A mechanism of degradation associated with layered structures is the phenomenon of debonding of adjacent layers known as "delamination". Delaminations can occur in structures constructed from layered materials such as "Advanced Composites", or may occur at the surface of a structure possessing a thin film coating. Such delaminations can occur during fabrication or as a consequence of low-velocity impact[1, 2]. When a structure containing delaminations is subjected to load, the layers adjacent to the delamination can buckle locally causing the debonded area to grow. Such delamination growth may have dramatic effects on the integrity of the corresponding structure, or cause flaking of a thin film coating thus reducing its effectiveness.

To date, several authors have examined various aspects of this problem[1–20] with the majority of papers being concerned with delamination associated with (initially) flat layers. Kachanov[4], however, considered the problem of a thin layer debonding from the interior surface of a cylindrical substrate which is subjected to external pressure and obtained bounds which correspond to a "critical stress" for initiation.

In the present work the problem corresponding to growth of a pre-existing delamination at the interface of a thin layer and a contracting cylindrical substrate is considered. The problem is approached as a moving boundary problem in the calculus of variations by employing the principle of stationary potential energy in conjunction with a Griffith type energy balance. A shallow arch theory is used as the mathematical model for the layer while the behavior of the substrate is assumed to be effectively unaltered by the presence of the layer and hence is modeled as a "rigid" but contracting "foundation". An analytical solution along with an associated growth criterion is found. Specific numerical results are obtained for both the case of a layer delaminating from the exterior surface of the substrate as well as for that of a layer debonding from the corresponding interior surface. These results reveal several characteristic features associated with the phenomena of interest and are seen in some cases to differ qualitatively from, and in others to be qualitatively similar to corresponding results obtained for flat layers.

FORMULATION OF THE PROBLEM

Consider the thin elastic film or layer adhered to the wall of a smooth circular cavity contained within a "rigid" but contracting "substrate" as shown in Fig. 1. The film is initially debonded from the substrate over a small portion of its surface and is shown in a buckled configuration. As the deflection of the cavity wall will be considered prescribed and uniform, we shall divide the film into three regions. The first, which shall be referred to as the "lift zone" or "lift region" of the film shall be defined over the domain $\mathscr{D}_1$, corresponding

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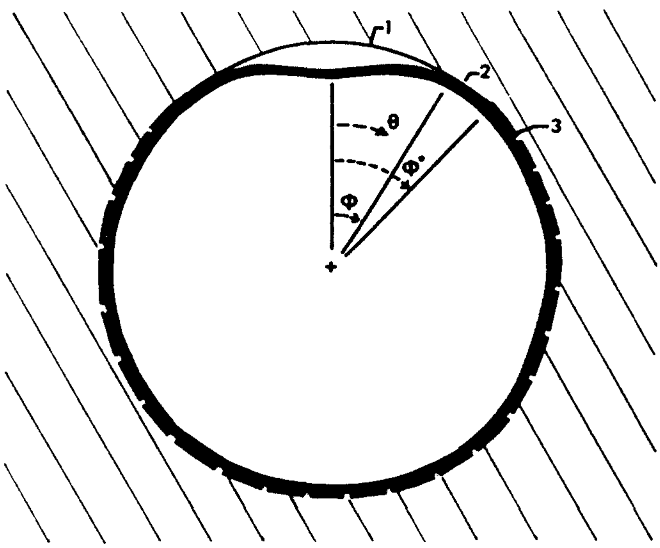

Fig. 1. Thin layer delaminating from the interior surface of a cylindrical substrate: (1) "lift zone", (2) "contact zone", (3) "bonded zone".

to $0 \leqslant \theta \leqslant \phi$ where θ is the angular coordinate measured clockwise from the vertical. It corresponds to the portion of the debonded segment of the film which is lifted away from the substrate. The second region of the layer will be referred to as the "contact zone" or "contact region" and corresponds to the portion of the debonded segment of the film in contact with the substrate. It is defined over the domain $\mathscr{D}_2$ corresponding to $\phi \leqslant \theta \leqslant \phi^*$. The final region of the film corresponds to the bonded portion of the film and is defined on the domain $\mathscr{D}_3$ which corresponds to $\phi^* \leqslant \theta \leqslant \pi$. Due to the symmetry of the problem we need only consider the half of the structure defined on $[0, \pi]$.

We shall approach the problem in the spirit of Bottega[7] and Bottega and Maewal[8–10] as a moving intermediate boundary problem in the calculus of variations with the intermediate boundary ϕ between the lift and contact zones and the intermediate boundary ϕ^* defining the size of the delamination being sought as part of the solution. The governing differential equations, associated boundary and interface conditions and traversality conditions resulting from the two moving intermediate boundaries are found by incorporating a Griffith type fracture criterion at the delamination edge and applying the theorem of stationary potential energy. The transversality condition at the delamination edge, $\theta = \phi^*$, yields a growth criterion governing the delamination while that at the "lift zone"/"contact zone" interface, $\theta = \phi$, establishes the conditions defining the lift zone. We shall assume that unbonded and debonded surfaces are smooth. The substrate will be assumed to be extremely "stiff" relative to the layer, hence the behavior of the substrate will be considered to be effectively unaltered by the presence of the layer. The contribution of the substrate to the energy release rate during growth of the delamination will therefore be neglected (see, e.g. Ref. [11]), thus allowing the problem to be formulated in terms of the energies associated with the debonding layer alone.

Let us begin by formulating the energy functional, Π, as follows:

$$\Pi = \sum_{i=1}^{3} U^{(i)} + \Lambda + \mathscr{E}_F \tag{1}$$

where $U^{(i)}$ corresponds to the normalized strain energy of the segment of the layer defined on $\mathscr{D}_i$, Λ is a constraint functional restricting the radial deflections on $\mathscr{D}_2$, and $\mathscr{E}_F$ is the normalized "fracture energy".

Within the context of the present formulation, the normalized strain energy of the segment of the layer defined on $\mathscr{D}_i$, $U^{(i)}$, will be composed of two parts $U_B^{(i)}$ and $U_M^{(i)}$ such that

$$U^{(i)} = U_{B}^{(i)} + U_{M}^{(i)} \quad (2)$$

where $U_B^{(i)}$ which corresponds to the normalized bending energy and $U_M^{(i)}$ which corresponds to the normalized membrane energy are given by

$$U_B^{(i)} = \frac{1}{2}\int_{\psi_a^{(i)}}^{\psi_b^{(i)}} K_i^2 \, d\theta \quad (3a)$$

and

$$U_M^{(i)} = \frac{1}{2C}\int_{\psi_a^{(i)}}^{\psi_b^{(i)}} N_i^2 \, d\theta \quad (3b)$$

where

$$\psi_a^{(1)} = 0, \qquad \psi_b^{(1)} = \phi \quad (4a, b)$$

$$\psi_a^{(2)} = \phi, \qquad \psi_b^{(2)} = \phi^* \quad (4c, d)$$

$$\psi_a^{(3)} = \phi^*, \qquad \psi_b^{(3)} = \pi \quad (4e, f)$$

and $C = 12/h^2$ is the normalized membrane stiffness of the layer while $h \ll 1$ is the normalized thickness of the layer. In eqns (3a) and (3b) K_i corresponds to the normalized "relative curvature" of the layer on $\mathscr{D}_i$ (the difference between the present curvature of the layer and the curvature of the layer in its undeformed configuration assumed equal to that of the initial curvature of the cavity wall) and N_i corresponds to the normalized resultant membrane force on $\mathscr{D}_i$. These quantities, which are variable on $\mathscr{S}_1 + \mathscr{S}_2$ and "prescribed" on $\mathscr{S}_3$, are given by

$$K_i = w_i'' + w_i \qquad (i = 1, 2) \quad (5a)$$

and

$$N_i = -C[u_i' - w_i + \tfrac{1}{2}w_i'^2] \qquad (i = 1, 2) \quad (5b)$$

$$K_3 = K_0 \equiv w_0/(1 - w_0) \approx w_0 \quad (5c)$$

and

$$N_3 \equiv P_0 = Cw_0 \quad (5d)$$

respectively, where we have employed the shallow arch equations of El-Bayoumy[21] which assume that $w_0 \ll 1$. In eqns (5a) and (5b), u_i (positive) clockwise and w_i (positive inward) correspond to the normalized circumferential and radial deformations, respectively, of a material particle located on the centerline of the segment of film on $\mathscr{S}_i$, while w_0 corresponds to the normalized deflection of the cavity wall, and a superposed prime denotes differentiation with respect to θ [i.e. $(\ )' \equiv d(\ )/d\theta$]. All deformations are normalized with respect to the initial radius of the cavity wall.

The constraint functional, Λ, which restricts the radial deflections of the segment of the layer in $\mathscr{S}_2$, is defined by

$$\Lambda = \int_{\phi}^{\phi^*} \lambda(w_2 - w_0) \, d\theta \quad (6)$$

where λ is a Lagrange multiplier, while the normalized "fracture energy", $\mathscr{E}_F$, is given by

$$\mathscr{E}_F = 2\gamma(\phi^* - \phi_0^*) \tag{7}$$

where γ is the normalized surface energy of the bond, a property of the layer material and substrate material, as well as the bond, and ϕ_0^* defines the initial size of the delamination.

For the problem at hand, the principle of stationary potential energy may be stated as

$$\delta\Pi = 0 \tag{8}$$

where δ corresponds to the variational operator and Π corresponds to the energy functional given by eqn (1).

Substitution of eqns (2)–(7) into (8) and taking into account the moving intermediate boundaries at $\theta = \phi$ and ϕ^* (see, e.g. Gelfand and Fomin[22]) yields the governing differential equations

$$\left.\begin{aligned} K_i'' + K_i + (N_i w_i')' + N_i &= p_i \quad (9)\\ N_i' &= 0 \quad (10)\end{aligned}\right\} \quad \theta \in \mathscr{D}_i \quad (i = 1,2)$$

and

$$w_2 = w_0, \qquad \theta \in \mathscr{D}_2 \tag{11}$$

where

$$p_1 = 0 \quad \text{and} \quad p_2 = \lambda \tag{12}$$

the boundary and matching conditions

$$[K_1' + N_1 w_1']_{\theta=0} = w_1'(0) = u_1(0) = 0 \tag{13a–c}$$

$$w_1(\phi) = w_2(\phi), \qquad w_1'(\phi) = w_2'(\phi), \qquad u_1(\phi) = u_2(\phi) \tag{14a–c}$$

$$N_1(\phi) = N_2(\phi) \tag{14d}$$

$$w_2(\phi^*) = w_0, \qquad w_2'(\phi^*) = u_2(\phi^*) = 0 \tag{15a–c}$$

and the transversality conditions at $\theta = \phi$ and ϕ^* resulting from the vanishing of coefficients of $\delta\phi$ and $\delta\phi^*$, respectively, in eqn (8). These conditions are given by

$$\left[\frac{1}{2}(K_1^2 - K_2^2) + K_1' w_1' - K_2' w_2' - K_1 w_1'' + K_2 w_2'' - \lambda(w_2 - w_0) + \frac{1}{2C}(N_1^2 - N_2^2) + N_1(u_1' + w_1'^2) - N_2(u_2' + w_2'^2)\right]_{\theta=\phi} = 0 \tag{16}$$

and

$$\left[\frac{1}{2}(K_2^2 - K_0^2) + K_2' w_2' - K_2 w_2'' + \frac{1}{2C}(N_2^2 - P_0^2) + \lambda(w_2 - w_0) + N_2(u_2' + w_2'^2) + 2\gamma\right]_{\theta=\phi^*} = 0. \tag{17}$$

From the form of eqns (9) it is apparent that p_2 may be identified as the contact pressure p, from which we have that

$$p_2 = p = \lambda, \qquad \theta \in \mathscr{D}_2. \tag{18}$$

If we next integrate eqn (10) and impose eqn (14d) we find that

$$N_1 = N_2 = N_0 = \text{constant} \tag{19}$$

where N_0 is still to be determined. Taking eqns (18) and (19) into account, the governing differential equations reduce to

$$\mathscr{L}\{w_i\} = p\delta_{i2} - N_0, \qquad \theta \in \mathscr{D}_i \qquad (i = 1, 2) \tag{20a, b}$$

and

$$w_2 = w_0, \qquad \theta \in \mathscr{D}_2 \tag{20c}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L} = \frac{d^4}{d\theta^4} + (N_0 + 2)\frac{d^2}{d\theta^2} + 1 \tag{21}$$

and δ_{ij} corresponds to Kronecker's delta. The associated boundary and matching conditions may be reduced to

$$w_1'''(0) = w'(0) = 0 \tag{22a, b}$$

$$w_1(\phi) = w_2(\phi), \qquad w_1'(\phi) = w_2'(\phi) \tag{23a, b}$$

$$w_2(\phi^*) = w_0, \qquad w_2'(\phi^*) = 0 \tag{24a, b}$$

with eqns (13c), (14c), (15c), and (20), combined to eliminate u_1 and u_2 and form an equivalent expression in terms of the functions w_1 and w_2 as follows:

$$\int_0^{\phi} (w_1 - \tfrac{1}{2}w_1'^2)\, d\theta + w_0(\phi^* - \phi) = \phi^* N_0/C. \tag{25}$$

The transversality conditions given by eqns (16) and (17) may similarly be reduced to the following forms:

$$w_1''(\phi) = w_2''(\phi) = 0 \tag{26}$$

and

$$\frac{1}{2C}(N_0 - P_0)^2|_{\theta = \phi^*} = 2\gamma \qquad (\phi < \phi^*) \tag{27a}$$

$$\left[\frac{1}{2}(K_1 - K_0)^2 + \frac{1}{2C}(N_0 - P_0)^2\right]_{\theta = \phi^*} = 2\gamma \qquad (\phi = \phi^*). \tag{27b}$$

Equations (26) and (27) define conditions for the values of the "lift zone"/"contact zone" interface ϕ, and the delamination boundary ϕ^*, respectively, such that each (ϕ, ϕ^*) pair corresponds to a state of equilibrium. As discussed below, each of the above conditions may be interpreted as an energy balance at the point in question.

The first transversality condition (26) is seen to specify that the location of the "lift zone"/"contact zone" interface, ϕ, which corresponds to an equilibrium configuration is one for which the bending moment is continuous across the interface.

The second transversality condition (27) states that the delamination size is such that the relative stretching energy at the delamination edge is balanced by the bond energy if the "lift zone" has not propagated throughout the delamination ($\phi < \phi^*$), or that the sum of the relative bending and relative stretching energies at the delamination boundary are balanced by the bond energy if the "lift zone" traverses the entire delamination. Equations (27) define the "delamination growth path" for the layer/substrate system. The following growth criterion is suggested by (27).

If

$$\mathcal{G}_a \equiv \frac{1}{2C}(N_0 - P_0)^2|_{\theta=\phi_0^*} \leqslant 2\gamma \qquad (\phi < \phi^*) \tag{28a-i}$$

or

$$\mathcal{G}_b \equiv \left[\frac{1}{2}(K_1 - K_0)^2 + \frac{1}{2C}(N_0 - P_0)^2\right]_{\theta=\phi_0^*} \leqslant 2\gamma \qquad (\phi = \phi^*) \tag{28a-ii}$$

no growth occurs with ϕ^* remaining at its initial value ϕ_0^*.

If

$$\mathcal{G}_a > 2\gamma \qquad (\phi < \phi^*) \tag{28b-i}$$

or

$$\mathcal{G}_b > 2\gamma \qquad (\phi = \phi^*) \tag{28b-ii}$$

growth occurs until ϕ^* satisfies eqns (27). The relative stretching energy in eqn (27a) and the sum of the relative bending and stretching energies in eqn (27b) correspond to the energy release rates during growth of the delamination. We thus see that the growth of the delamination is governed by either mode II fracture or a combination of mode I and mode II fracture. We may also note that, within the context of the present model, growth does not occur unless local buckling of the layer occurs.

The system of differential equations, eqns (20), along with conditions (22)–(27), constitute a multiple moving boundary problem for the deflections $w_1(\theta)$, the membrane force N_0, the "interface angle" ϕ, and the delamination boundary ϕ^*. The corresponding analytical solution will be presented in the next section.

ANALYTICAL SOLUTION

We shall now obtain the analytical solution corresponding to the problem formulated in the previous section. The solution shall correspond to two phases, the first being when the "lift zone" is smaller than or just equal to the size of the delamination with growth governed by eqns (28a-i) and (28b-i). The second phase is such that the "lift zone" traverses the entire delamination ($\phi = \phi^*$) and growth is governed by eqns (28a-ii) and (28b-ii). The phase corresponding to $\phi < \phi^*$ shall be considered first.

Equation (20a) can be solved for $w_1(\theta)$ to yield

$$w_1(\theta) = (w_0 + N_0)F(\theta; \alpha, \phi) - N_0 \tag{29}$$

where

$$F(\theta;\alpha,\phi)\equiv\frac{\left[\tan\phi/\alpha\,\dfrac{\cos\alpha\theta}{\cos\alpha\phi}-\alpha^2\tan\alpha\phi\,\dfrac{\cos\theta/\alpha}{\cos\phi/\alpha}\right]}{[\tan\phi/\alpha-\alpha^2\tan\alpha\phi]} \tag{30}$$

and

$$\alpha^2=\tfrac{1}{2}[N_0+2+\sqrt{(N_0(N_0+4))}]>1\quad\text{or}\quad N_0=(\alpha^2-1)^2/\alpha^2 \tag{31}$$

which when substituted into eqns (26) and (27) give the explicit forms of the transversality conditions. We thus have that

$$\tan\alpha\phi-\alpha^2\tan\phi/\alpha=0 \tag{32}$$

and either

$$\frac{1}{2C}(N_0-Cw_0)^2=2\gamma\qquad(\phi<\phi^*) \tag{33a}$$

or

$$\tfrac{1}{2}[(w_0+N_0)F''-w_0K_0]^2|_{\theta=\phi^*}+\frac{1}{2C}(N_0-Cw_0)^2=2\gamma\qquad(\phi=\phi^*). \tag{33b}$$

As seen by eqn (20c), the radial deflection of the layer in the "contact zone" $\mathscr{D}_2$ is equal to that of the substrate wall, and when substituted into (20b) gives

$$p=\lambda=w_0+N_0\qquad(\phi\leqslant\theta\leqslant\phi^*).$$

Substitution of eqn (29) into eqn (25) gives the required condition

$$(w_0+N_0)H(\alpha,\phi)-\tfrac{1}{2}(w_0+N_0)^2Z(\alpha,\phi)+w_0(\phi^*-\phi)-N_0\left(\phi+\frac{\phi^*}{C}\right)=0 \tag{34}$$

where

$$H(\alpha,\phi)=-\frac{(\alpha^4-1)}{\alpha D}\tan\alpha\phi\tan\phi/\alpha \tag{35a}$$

$$\begin{aligned}Z(\alpha,\phi)=\;&\frac{\alpha\tan^2\phi/\alpha}{2D^2\cos^2\alpha\phi}[\alpha\phi-\cos\alpha\phi\sin\alpha\phi]\\&+\frac{\alpha^3\tan^2\alpha\phi}{2D^2\cos^2\phi/\alpha}[(\phi/\alpha)-\cos\phi/\alpha\sin\phi/\alpha]\\&-\frac{2\alpha^3}{(\alpha^4-1)}\frac{\tan\alpha\phi\tan\phi/\alpha}{D^2}[\tan\alpha\phi-\alpha^2\tan\phi/\alpha]\end{aligned} \tag{35b}$$

and

$$D=\tan\phi/\alpha-\alpha^2\tan\alpha\phi. \tag{35c}$$

Equation (34) may be solved for w_0 explicitly, upon expanding and neglecting terms of $O(w_0^2)$ compared with those of $O(w_0)$. We then have

$$w_0 = w_0(\alpha, \phi, \phi^*) = \frac{N_0(\phi+\phi^*/C) - N_0[H(\alpha,\phi) - \frac{1}{2}N_0 Z(\alpha,\phi)]}{H(\alpha,\phi) - N_0 Z(\alpha,\phi) + (\phi^* - \phi)}. \quad (34')$$

Equations (32), (33a), and (34) constitute a system of coupled non-linear algebraic equations which may, in principle, be solved simultaneously to yield N_0, ϕ, and ϕ^* as a function of w_0 for given values of C and γ.

Once the lift zone completely envelops the debonded segment of the layer, conditions (32) and (33a) are replaced by eqn (33b) (a condition is removed since it is established that $\phi = \phi^*$) as lifting may continue without ϕ increasing. Equations (33b) and (34) may be solved simultaneously to give N_0 and ϕ^* as a function of w_0 for given values of the normalized membrane stiffness C and normalized bond energy γ.

NUMERICAL RESULTS

We shall next present results corresponding to small contractions of the substrate boundary (i.e. $w_0 \ll 1$) for the specific problem where $C = 2 \times 10^6$. The results will be given for the case of a layer delaminating from the exterior surface of a cylindrical substrate as well as for the case of a layer delaminating from the interior surface of a cylindrical substrate (or equivalently from the wall of a cylindrical cavity). The latter will be considered first.

Interior layer

As seen from eqns (27) and (28), growth of the delamination is dependent upon the relative stretching energy at the delamination edge, $\mathcal{G}_a$, until the lift zone just reaches the delamination boundary. After this occurs, growth of the delamination is governed by the sum of the relative stretching and relative bending energies at the delamination tip, $\mathcal{G}_b$.

We shall first consider the case where $\phi \leqslant \phi^*$. As described above, substitution of eqn (34′) into eqn (33a) in conjunction with eqn (32) results in a non-linear algebraic equation in α, ϕ, and ϕ^*, from which an infinite number of roots, α, may be found for each combination of ϕ and ϕ^*. We shall seek the lowest value of α (i.e. the value of α associated with the "first buckling mode") for which the "relative crown point deflection", $\Delta_0 \equiv w(0) - w_0$, is positive. The latter may be found by evaluating eqn (29) at $\theta = 0$ and then subtracting the substrate wall deflection, w_0, which may be obtained from eqn (34′) for the specific (α, ϕ, ϕ^*) combination.

The roots, α, of the aforementioned non-linear algebraic equation are found for specific delamination boundary angles, ϕ^*, for desired "lift zone"/"contact zone" interface angles, $\phi \leqslant \phi^*$, by first iterating on α and finding intervals upon which the left-hand side of the equation changes sign and then by employing the bisection technique (interval halving). The resulting cavity wall contractions, w_0, and "relative crown point deflections", Δ_0, are then found as discussed earlier, and the energy release rates, $\mathcal{G}_a$, are subsequently evaluated. The computed values of $\mathcal{G}_a$ are displayed in Fig. 2 as a function of w_0 for various values of the delamination boundary angle ϕ^*.

Each point lying on a curve displayed in Fig. 2 corresponds to a specific value of ϕ, with ϕ increasing as we traverse the curve in a clockwise fashion, the curve terminating at the point which corresponds to $\phi = \phi^*$. We note here that as $\phi \to 0$, w_0 increases, thus prohibiting the curve from intersecting $w_0 = 0$. These "leftward progressing" portions of each curve correspond to unstable values of the "interface angle"† and would likely bend toward the origin in the presence of geometrical imperfections.‡ We shall therefore consider only the "rightward progressing" (i.e. "upper") portions of these curves. Though the precise points at which these curves are intersected as w_0 increases are unknown, we shall assume, for later discussions, that such intersection points are near the leftmost portion of the curve, i.e. the point corresponding to the minimum value of w_0 on that curve. With this assumption we see that "small" delaminations effectively "exist" only with $\phi = \phi^*$.

Upon further observation of Fig. 2, we note that for a given value of $\mathcal{G}_a$, w_0 decreases as ϕ^* increases with the corresponding points getting closer and closer as we increase ϕ^*.

† See, e.g. Ref. [21], where the special case of a completely debonded layer was considered.
‡ See, e.g. Ref. [23].

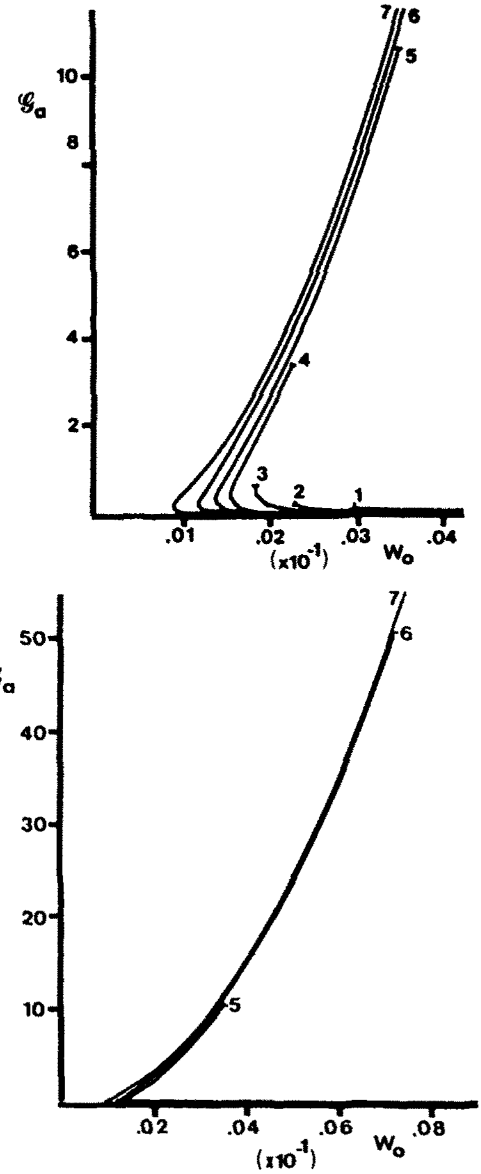

Fig. 2. Relative stretching energy at delamination edge vs substrate wall contraction for various delamination sizes: (1) $\phi^* = 0.062$, (2) $\phi^* = 0.074$, (3) $\phi^* = 0.10$, (4) $\phi^* = 0.15$, (5) $\phi^* = 0.20$, (6) $\phi^* = 0.30$, (7) $\phi^* = 0.60$. The "lift zone"/"contact zone" interface angle $\phi \leqslant \phi^{*-}$ increases along each path as the path is traversed in a clockwise fashion ($C = 2.0 \times 10^6$).

This implies that for a given value of $\mathcal{G}_a$, w_0 decreases approaching an asymptotic value as ϕ^* increases. As lower values of w_0 correspond to lower values of the energy of the layer, for a given mode, the above, along with eqns (28a-i) and (28b-i), imply that when growth occurs, (and $\phi \leqslant \phi^{*-}$) it is extensive. Lastly, it is seen from Fig. 2 that for a given delamination size, ϕ^*, the maximum relative stretching energy at the delamination tip occurs when the lift zone spans the entire delamination. In other words, for a given ϕ^*, $\mathcal{G}_a = \mathcal{G}_{a_{max}}$ when $\phi = \phi^{*-}$. These values are plotted as a function of delamination angle and are presented in Fig. 3. From the growth criterion (28) it is seen that the curve displayed in Fig. 3 defines the "boundary" separating the "regions" where $\mathcal{G}_a$ governs initial delamination growth (below the curve) from the regions where $\mathcal{G}_b$ initially governs delamination growth (above the curve). In addition, we note from Fig. 3 that the curve is monotonically increasing from which it may be inferred that in a general sense, $\mathcal{G}_a$ governs the growth of "large" delaminations while $\mathcal{G}_b$ governs the growth of "small" delaminations.

Once the "lift zone" spreads through the entire span of the delamination, the curvature of the layer is no longer restricted at any point to be that of the substrate wall. For this

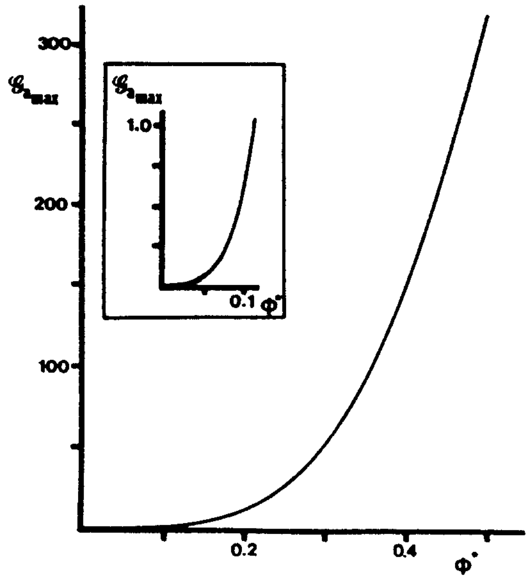

Fig. 3. Variation of maximum relative stretching energy with delamination size ($C = 2.0 \times 10^6$).

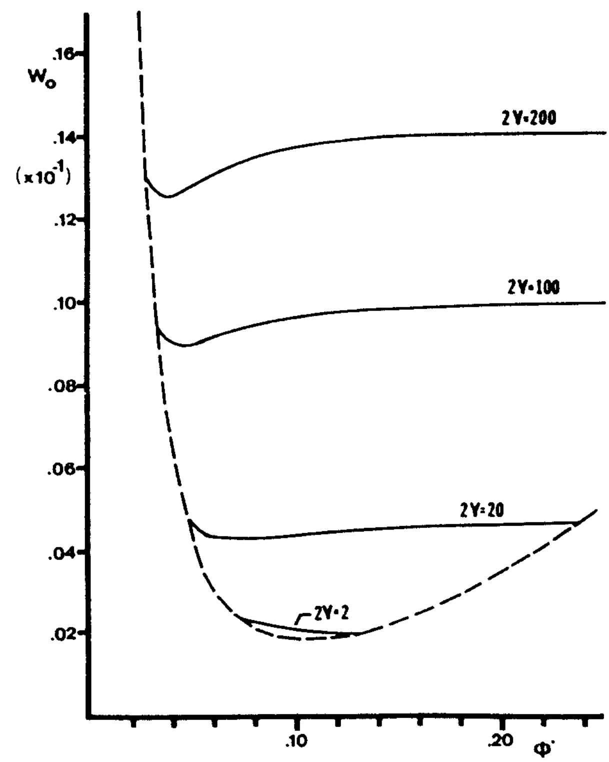

Fig. 4. Delamination growth paths corresponding to a thin layer adhered to the interior surface of substrate when the "lift zone" spans the entire delamination (i.e. when $\phi = \phi^*$), and $C = 2.0 \times 10^6$.

case $\mathcal{G}_b$ governs growth and solution (29) together with the associated transversality condition (33b), corresponding to the case where $\phi = \phi^*$, now holds. In this regard, upon substitution of eqn (34′), eqn (33b) may be solved numerically for given values of γ and C, in the manner described above, to yield the lowest admissible $\alpha > 0$ for each ϕ^*. The corresponding values of w_0, subsequently obtained from eqn (34′), may be plotted vs ϕ^* to yield the "delamination growth paths" associated with a given bond strength, 2γ. Four such "paths", corresponding to the cases $\gamma = 1$, 10, 50 and 100, are shown in Fig. 4. The dashed curve in the figure corresponds to the value of w_0 at each ϕ^* such that $\mathcal{G}_a = \mathcal{G}_{max}$. We note that the curves appear asymptotic as ϕ^* increases, indicating that if a substrate deflection greater than a certain value is achieved before growth begins, growth will be unstable and extensive (catastrophic) once it does begin. Such phenomena is dependent upon the initial delamination size, and occurs when $\phi_0^* \lesssim 0.050$ for $\gamma = 10$, $\phi_0^* \lesssim 0.030$ for $\gamma = 50$ and when $\phi_0^* \lesssim 0.027$ for $\gamma = 100$. Growth is seen to be catastrophic for all values of ϕ_0^* corresponding to $\gamma = 1$. We also note the existence of a critical delamination size, $\phi^* = \phi_c^*$, corresponding to the "minimum" on each "path". Such a point separates the occurrence of "stable growth" corresponding to an incremental increase in delamination size with an incremental increase of w_0 when $\phi^* > \phi_c^*$, from the occurrence of "unstable growth" when $\phi^* < \phi_c^*$ where the delamination size increases a finite amount with an incremental increase in w_0 and is then followed by stable growth. Such values of ϕ^* are bounded by the catastrophic regions defined above before growth begins. Once growth begins, the corresponding path is followed. For the cases considered we see that $\phi_c^* \approx 0.070$ for $\gamma = 10$, $\phi_c^* \approx 0.046$ for $\gamma = 50$, and $\phi_c^* \approx 0.038$ for $\gamma = 100$. Let us consider the following specific examples. Suppose first that a delamination of angle $\phi^* = 0.30$ exists at a layer/substrate interface with bond strength $2\gamma = 20$ (as before, $C = 2 \times 10^6$). We note from Fig. 3 that $\mathscr{G}_a = 20$ lies below the curve and hence that $\mathscr{G}_a$ governs growth for this case. If the cylinder is subsequently contracted, w_0 increases and eventually intersects the path in Fig. 2 corresponding to $\phi^* = 0.30$. As w_0 is further increased ϕ increases (recall that ϕ increases as we progress clockwise along the corresponding paths in Fig. 2) until $w_0 \approx 0.46 \times 10^{-1}$ at which point $\mathscr{G}_a = 2\gamma$. From the above discussion we see that at this point delamination growth occurs in an unstable and catastrophic manner.

Let us next consider a delamination of initial size $\phi_0^* = 0.1$ for bond strengths of $2\gamma = 2$ and 100. We see from Fig. 3 that $\mathscr{G}_b$ governs growth for both cases and we see from Fig. 2 that for a bond strength of $2\gamma = 2$, $\phi = \phi^*$ almost immediately after the curve is intersected corresponding to a value of $w_0 \approx 0.00185$. For this same bond strength, it is seen from Fig. 4 that catastrophic growth begins when w_0 is increased slightly to a value of $w_0 \approx 0.00205$. For a bond strength of $2\gamma = 100$, however, it is seen that no growth occurs until $w_0 \approx 0.0097$ with subsequent increases in w_0 resulting in stable growth of the delamination. Finally, let us consider a delamination of initial size $\phi_0^* = 0.062$ at an interface with bond strength $2\gamma = 20$. We note from Fig. 3 that $\mathscr{G}_b$ governs growth. It is seen from Fig. 4 that as w_0 is increased, no growth occurs until $w_0 \approx 0.00435$ at which time the delamination grows in an unstable manner until it reaches a size of $\phi^* \approx 0.09$. Further increases in w_0 result in a stable spreading of the delamination until ϕ^* grows to a value of approximately 0.228 at which point $\mathscr{G}_a$ governs once again and catastrophic growth occurs.

In general then, we see that, as discussed earlier, delamination growth of an interior layer may occur when $\phi < \phi^*$ (via "mode II fracture") and that when such growth occurs it occurs in an "unstable" and "catastrophic" manner. We also see that ϕ may progress with increasing w_0 unaccompanied by delamination growth until $\phi = \phi^*$. If this occurs, growth may occur via a combination of "mode I" and "mode II" fracture and may progress in a "stable" manner, in an "unstable" manner, or in an "unstable" and "catastrophic" manner.

Exterior layer

We next consider results corresponding to a layer which is delaminating from the exterior surface of a cylindrical substrate. For this problem the layer buckles away from the contracting substrate as shown in Fig. 5 with no contact with the substrate except, of course, at $\phi = \phi^*$. Solution (29) with transversality condition (33b) is therefore applicable if we consider solutions such that the "relative crown point deflection" is negative, and interpret all quantities as being normalized with respect to the exterior radius of the substrate. Proceeding as for the previous case we obtain the growth paths corresponding to the lowest value of α, for each ϕ^*, such that $\Delta_0 < 0$.

The growth paths corresponding to $C = 2 \times 10^6$ and $\gamma = 1, 10, 50$, and 100 are displayed in Fig. 6. It is seen that each path contains a minimum with the corresponding value of ϕ^* specifying a critical delamination size which separates delamination sizes that grow in a stable manner from those which grow in an unstable manner. It is also seen that each path

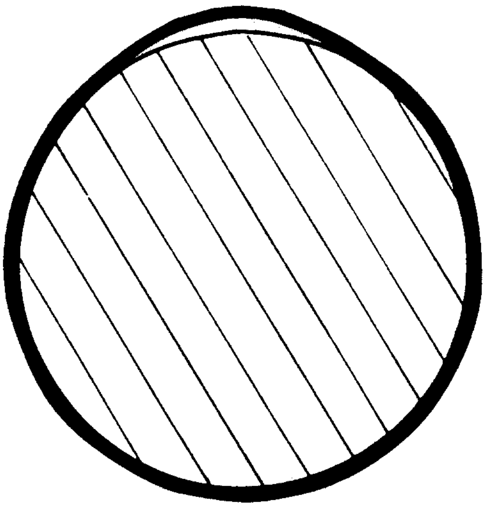

Fig. 5. Thin layer delaminating from the exterior surface of a cylindrical substrate.

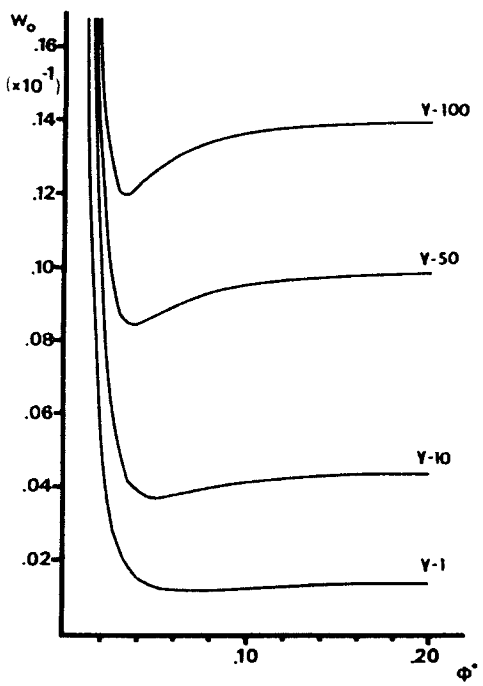

Fig. 6. Delamination growth paths corresponding to a thin layer adhered to the exterior surface of a cylindrical substrate ($C = 2.0 \times 10^6$).

tends to an asymptote indicating the possibility of "catastrophic growth". As the left-hand portions of the growth paths increase with decreasing ϕ^* we note the existence of a second "critical delamination size" such that delaminations smaller than this size grow catastrophically once growth begins. Finally we note, upon observing each path in Fig. 6, that as the bond energy decreases, the "stable well" of the path widens and its depth decreases indicating a tendency toward more extensive unstable growth and catastrophic behavior with decreasing bond energy. For the specific delamination growth paths shown, we see that unstable growth occurs, followed by stable growth, for delaminations which are such that $\phi^* \lesssim 0.032$ for $\gamma = 100$, $\phi^* \lesssim 0.038$ for $\gamma = 50$, $\phi^* \lesssim 0.051$ for $\gamma = 10$, and $\phi^* \lesssim 0.067$ for $\gamma = 1$, while catastrophic growth occurs for those same values of the bond energy, when $\phi^* \lesssim 0.022$, 0.025, 0.034, and 0.044, respectively.

As an example let us consider the case of an initial delamination, of size $\phi_0^* = 0.030$, exists at the interface between an exterior layer and a cylindrical substrate where the normalized energy of the bond is given by $2\gamma = 100$ ($\gamma = 50$). As the cylinder contracts, no growth occurs until $w_0 \approx 0.00885$. The delamination then grows in an unstable manner until $\phi^* \approx 0.054$. Further increases in the substrate wall deflection result in stable growth of the delamination as we follow the rising portion of the growth path shown. If we consider a delamination of initial size $\phi_0^* = 0.022$, no growth occurs until the cylinder wall contracts to $w_0 \approx 0.01275$ at which time growth occurs catastrophically. Finally, if we consider a delamination of initial size $\phi_0^* = 0.060$ we observe that as we contract the cylinder, no growth occurs until $w_0 \approx 0.009$ at which time growth begins with the delamination size increasing in a stable manner as the cylinder wall is further contracted.

To conclude then, we see that delamination growth at the interface of layers bonded to the exterior surface of the substrate occurs via a combination of "Mode I" and "Mode II" fracture, with the qualitative features of the growth behavior more closely resembling those for flat layers than do those of the interior layers (see, e.g. Refs [6, 8]).

CONCLUDING REMARKS

The problem of a thin layer delaminating from a contracting cylindrical substrate was considered for the case where the equilibrium of the substrate is effectively unaltered by the behavior of the surface layer. The problem was formulated as a moving boundary problem in the calculus of variations where the "moving boundaries" corresponding to the "lift zone"/"contact zone" interface, and the delamination edge, were parameterized by their corresponding angles, while a shallow arch theory was employed as the mathematical model for the layer. The "strength" of the bond was characterized by the surface energy of the bond, a property of the layer/substrate/bond system.

The resulting transversality conditions were seen to specify the criteria for the "interface" and "delamination angles" to correspond to a state of equilibrium. It was shown that the lift zone/contact zone interface angle must be such that the bending moment is continuous at that point. The criteria governing delamination growth resulted from the employment of a Griffith type criterion in the energy functional, from which it followed that if a portion of the delaminated segment of the layer considered is in contact with the substrate, the corresponding growth of the delamination is governed by the "relative stretching energy" at the delamination edge, with the associated delamination angle corresponding to a state of equilibrium being such that the aforementioned energy is balanced by the energy of the bond. It was also found that if the "life zone" traverses the entire delamination prior to the onset of growth, then the growth of the delamination is governed by the sum of the "relative stretching" and "relative bending energies" at the delamination edge, with the value of the delamination angle which corresponds to a state of equilibrium defined in a manner analogous to that for the previous case.

Numerical results corresponding to specific values of the normalized layer thickness and bond energy were obtained for both the case of a layer delaminating from the exterior surface of the substrate and the case of a layer delaminating from the interior surface of the substrate. Growth behavior predicted at the interface between an exterior layer and the substrate was seen to qualitatively resemble that predicted by Chai *et al.*[6] for the analogous problem of a flat layer on a substrate, while growth behavior at the interface between an interior layer and the substrate was seen to be substantially more complicated. This "complication" is seen to be due to the existence, and size change, of the "contact zone" arising from the non-vanishing curvature of the substrate. Delamination growth predicted for both exterior and interior interfaces was seen to occur in either a stable manner, an unstable manner followed by a stable manner, or in a catastrophic manner. It was observed, however, that predicted growth at the interior interface exhibited a greater tendency toward catastrophic growth.

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